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UNIFYING SIGMOID UNIVARIATE GROWTH EQUATIONS

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ABSTRACT. An equation depending on two shape parameters includes many of the sigmoids found in the literature as special cases. The formulation facilitates the analysis and description of model properties, and can be used to enhance the generality of software and statistical procedures. Some of the models obtained may be new. The relationship between growth curves and differential equations is discussed; the non-uniqueness of the differential forms has not been generally recognized.

Keywords: Growth and yield models, growth curves, differential equations, Burr distribution, Box-Cox transformation, Richards, logistic.

1 INTRODUCTION

Many univariate growth and yield models have been used in forestry. Compilations of these are given, among others, by Peschel (1938), Grosenbaugh (1965), Zeide (1993), and Kiviste et al. (2002). Although by no means the only important model type, attention has focused on sigmoid yield curves, with an inflexion point and a horizontal upper asymptote, and on their differential (growth) forms.

It can be useful to have "generalized" equations, which contain others as special cases. For instance, for particular values of its parameters or as limiting cases, Richards' model includes the monomolecular, Gompertz, and logistic (von Bertalanffy, 1949; Richards, 1959). A general model can simplify the study of properties such as the presence and nature of asymptotes and inflection points, and facilitate the development of more widely applicable software. I examine here an expression that includes many of the known univariate sigmoid growth and yield models. To simplify, we ignore any linear transformations of the size y and time t, and assume that y has been scaled to the interval $0 \leq y \leq 1$. Thus, disregarding differences in location and scale, the various models correspond to specific values of two "essential" shape parameters.

Some preliminaries on variable transformations are presented in the next Section, followed by the general model in its integrated (yield) form $y = f(t)$. Differential forms are addressed after that. Finally, admissible parameter ranges and the location of the inflection point are investigated, and the results are summarized. Some readers may benefit from reading the Summary first.

Note that the quest for yield equations is essentially the same as that for probability distributions having a closed form formula for the cumulative (Burr, 1942). In fact, the general yield model given here is related to some of Burr's distributions.

2 THE BOX-COX TRANSFORMATION

Most growth and yield equations contain powers, logarithms, or exponentials. These functions can conveniently be treated together, observing that

$$\lim_{c \to 0} \frac{x^c - 1}{c} = \ln x \ .$$

The expression under the limit is a transformation proposed by Box and Cox (1964), often used to approximate linearity and/or normality in statistical models. García (1983), for example, found that with it the Richards growth model and its special cases reduce to a simple linear differential equation. Nelder et al. (1960), Nelder (1961, 1962), obtained the Richards from a power transformation of the logistic.

For our purposes it will be convenient to use the negative

$$B(x,c) = \begin{cases} \frac{1-x^c}{c} & \text{if } c \neq 0 \ , \\ -\ln x & \text{if } c = 0 \ , \end{cases} \tag{1}$$

or, more compactly,

$$B(x,c) = \lim_{p\to c} \frac{1-x^p}{p} \ , \tag{2}$$

where $x \geq 0$.

The inverse transformation is

$$B^{-1}(x,c) = \lim_{p\to c}(1-px)^{1/p} \ , \tag{3}$$

or

$$B^{-1}(x,c) = \begin{cases} (1-cx)^{1/c} & \text{if } c \neq 0 \ , \\ \mathrm{e}^{-x} & \text{if } c = 0 \ , \end{cases} \tag{4}$$

defined for $cx \leq 1$.

For future reference, note here the derivative

$$\frac{\mathrm{d}B(x,c)}{\mathrm{d}x} = -x^{c-1} \ , \tag{5}$$

and the limits

$$\lim_{x\to\infty} B^{-1}(x,c) = 0 \qquad \text{for } c \leq 0, \tag{6}$$

$$\lim_{x\to-\infty} B^{-1}(x,c) = \infty \qquad \text{for } c \geq 0. \tag{7}$$

The limits do not exist for c outside the specified ranges.

3 THE GENERAL YIELD MODEL

Most univariate yield models express a power or logarithmic function of y in terms of power or exponential functions of t. Using the Box-Cox transformation (1)–(2), consider

$$B(B(y,a),b) = t, \tag{8}$$

where a and b are parameters, $0 \leq y \leq 1$, and there may be some linear transformation implicit in t. Solving for y gives the yield equation

$$y = B^{-1}(B^{-1}(t,b),a) \ , \tag{9}$$

or

$$y = \lim_{p\to a, q\to b}[1-p(1-qt)^{1/q}]^{1/p} \ . \tag{10}$$

More explicitly,

$$y = \begin{cases} [1-a(1-bt)^{1/b}]^{1/a} & \text{if } a,b \neq 0, \\ \exp[-(1-bt)^{1/b}] & \text{if } a=0,\ b\neq 0, \\ (1-a\mathrm{e}^{-t})^{1/a} & \text{if } a\neq 0,\ b=0, \\ \exp(-\mathrm{e}^{-t}) & \text{if } a,b=0. \end{cases}$$

As a shorthand, write $G(a,b)$ for the generalized model (9)–(10). Write also $\overline{G}(a,b)$ for the same relationship, but with $1-y$ in place of y, and t multiplied by -1. That is, with the ranges of y and t reversed;

$$\overline{G}(a,b): \quad 1-y = B^{-1}(B^{-1}(-t,b),a) \ .$$

With this notation we can limit the linear transformations in y and t to positive scale factors and a horizontal shift.

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It can be verified that, up to linear transformations, (9)–(10) includes all the models listed by Zeide (1993), with the single exception of Sloboda's equation $y = a\exp[-b\exp(-ct^d)]$. Of those, the most general are the Levakovic I,

$$y = (\frac{t^d}{1+t^d})^c \quad \longleftrightarrow \quad G(-1/c, -1/d) \ ,$$

Richards,

$$y = (1 - \mathrm{e}^{-t})^c \quad \longleftrightarrow \quad G(1/c, 0)$$

or

$$y = (1 + \mathrm{e}^{-t})^{-c} \quad \longleftrightarrow \quad G(-1/c, 0) \ ,$$

Korf,

$$y = \exp(1/t^c) \quad \longleftrightarrow \quad G(0, -1/c) \ ,$$

and Weibull,

$$y = 1 - \exp(-t^c) \quad \longleftrightarrow \quad \overline{G}(0, 1/c) \ ,$$

where the parameters c and d are positive. The other models are special instances of these (with Zeide's terminology and notation): Hossfeld IV and Yoshida I $\leftrightarrow G(-1, -1/c)$, Gompertz $\leftrightarrow G(0, 0)$, Logistic $\leftrightarrow G(-1, 0)$, Monomolecular $\leftrightarrow G(1, 0)$, Bertalanffy $\leftrightarrow G(1/3, 0)$, Levakovic III $\leftrightarrow G(-1/c, -1/2)$. As will be seen, some apparently new extensions in the range of the parameters are possible.

Of the Burr distributions over $(0, \infty)$, the Type III is $G(-1/k, -1/c)$ (same as the Levacovic I equation), and the more commonly used Burr Type XII is $\overline{G}(-1/k, 1/c)$, for $c, k > 0$ (Burr, 1942; Rodríguez, 1977; Tadikamalla, 1980).

4 DIFFERENTIAL FORMS

Yield can be thought of as accumulated growth. That is to say, a yield equation is the integral of a differential equation expressing the growth rate. Peschel (1938),Grosenbaugh (1965),Zeide (1993), and Kiviste et al. (2002), give one differential equation for each of the various yield models. Contrary to what that might imply, however, the differential equation is not unique: many different growth equations can generate the same yield curve.

Given $y = f(t)$, a differential equation can be obtained as $\mathrm{d}y/\mathrm{d}t = f'(t)$. This is a function of t. But one can also substitute $t = f^{-1}(y)$ to obtain $\mathrm{d}y/\mathrm{d}t = f'[f^{-1}(y)]$, a differential equation depending only on y. The same result can often be produced more easily by differentiating $t = f^{-1}(y)$ (example below). Many growth equations presented in the literature contain both y and t, having been derived through differentiating various transformations of the yield equation. For instance, differentiating both sides of $\ln y = \ln f(t)$ gives $(1/y)\,\mathrm{d}y = f'(t)/f(t)\,\mathrm{d}t$, from where $\mathrm{d}y/\mathrm{d}t = yf'(t)/f(t)$. In fact, the alternatives are infinite, as can be shown by substituting $y^\alpha[f(t)]^{1-\alpha}$ for y, where α is any real number.

Does it matter? And if so, which form should be used? With a fixed origin, and in a deterministic world, integrating any of these growth equations results in exactly the same yield. However, there are differences when treatments or random variation alter the trajectories (García, 1983, 2005). Then it can be important to decide which variable(s) better determine the growth rate: age and/or size. There are usually good theoretical reasons to prefer y over the elapsed chronological time t, especially in plants. A significant effect of age, independently and in addition to size, may point to the existence of hidden state variables, and to the desirability of a multivariate model (García, 1994).

The general age-invariant growth equation can be obtained from (8):

$$\frac{\mathrm{d}B(B(y,a),b)}{\mathrm{d}t} = 1 \ . \tag{11}$$

Differentiating,

$$B'(B(y,a),b)\frac{\mathrm{d}B(y,a)}{\mathrm{d}t} = 1 \ ,$$

$$\frac{\mathrm{d}B(y,a)}{\mathrm{d}t} = -B(y,a)^{1-b} \ , \tag{12}$$

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$$\frac{\mathrm{d}y}{\mathrm{d}t} = y^{1-a} B(y, a)^{1-b} \ . \tag{13}$$

Or

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \begin{cases} y^{1-a}(\frac{1-y^a}{a})^{1-b} & \text{if } a \neq 0, \\ y(-\ln y)^{1-b} & \text{if } a = 0. \end{cases}$$

Instead of (8)–(10), equations (11), (12) or (13) could be taken as the general model definition. Then the yield curve equations are obtained by integration with appropriate initial conditions. With (t_0, y_0) we obtain from (11)

$$B(B(y, a), b) = B(B(y_0, a), b) + t - t_0 \ . \tag{14}$$

5 PROPERTIES

A sigmoid must have an upper asymptote at $y = 1$, that is, $\lim_{t\to\infty} f(t) = 1$. From (9) and (6), the asymptote of $G(a, b)$ exists only for $b \leq 0$. Similarly, using both (6) and (7) it is found that $\overline{G}(a, b)$ has an upper asymptote if, and only if, $a \leq 0$ and $b \geq 0$. It can also be verified that limits as $t \to -\infty$ are 0 or do not exist, so that we are justified in restricting the t-scale factor in $G(a, b)$ and $\overline{G}(a, b)$ to be positive.

The inflection point is where the slope reaches a maximum. Differentiating (13) and equating to zero, it is found that the y-position of the inflection point of $G(a, b)$, if it exists, is

$$y_{\text{infl}} = \begin{cases} (\frac{1-a}{1-ab})^{1/a} & \text{if } a \neq 0, \\ \mathrm{e}^{b-1} & \text{if } a = 0. \end{cases} \tag{15}$$

From symmetry, the inflection point of $\overline{G}(a, b)$ is located at $1 - y_{\text{infl}}$. Examining the conditions under which there is an asymptote, and also an inflection point between 0 and 1, it is concluded that $G(a, b)$ is a sigmoid for

$$a < 1 \ , \quad b \leq 0 \ , \quad \text{and} \quad ab < 1 \ , \tag{16}$$

and $\overline{G}(a, b)$ is a sigmoid for

$$a \leq 0 \quad \text{and} \quad 0 \leq b < 1 \ . \tag{17}$$

Additional conditions are required for the yield curve to have an origin, that is, to have $y = 0$ for some finite $t = t_0$. For $G(a, b)$, (8) shows that

$$t_0 = B(B(0, a), b) = \begin{cases} B(1/a, b) & \text{if } a > 0, \\ 1/b & \text{if } a \leq 0 \text{ and } b < 0, \\ -\infty & \text{if } a \leq 0 \text{ and } b \geq 0. \end{cases}$$

Within the acceptable range $b \leq 0$, $G(a, b)$ does not go through an origin when $b = 0$ and $a \leq 0$. Well-known examples are the Gompertz, $G(0, 0)$, and the logistic, $G(-1, 0)$. For $\overline{G}(a, b)$, the origin would be at $t_0 = -B(B(1, a), b) = -B(0, b)$, which is finite if $b \neq 0$.

Contour curves for the height of the inflection point can be drawn by solving (15) for b:

$$b = [1 - (1 - a)/y_{\text{infl}}^a]/a \ .$$

These are shown in Figure 1.

The $G(a, b)$ sigmoid region (16) contains the Levakovic I, which is defined with $a < 0$ and $b < 0$ (Zeide, 1993). Other special cases are indicated in Figure 1. I am not aware of any published models with $a > 0$ and $b < 0$.

From the $\overline{G}(a, b)$ family, only the Weibull, $a = 0$, seems to have been proposed as a growth model in the literature (Yang et al., 1978)[1]. The more general form corresponds to the Burr Type XII probability distribution (Burr, 1942; Rodríguez, 1977; Tadikamalla, 1980).

[1] The nonlinear least-squares fits of the Bertalanffy-Richards model in Yang et al. (1978), however, failed to converge. Recalculation with their test data sets gives smaller residual sums of squares for this model than for the Weibull. Anyway, it seems doubtful that tree volume growth would be asymptotic.

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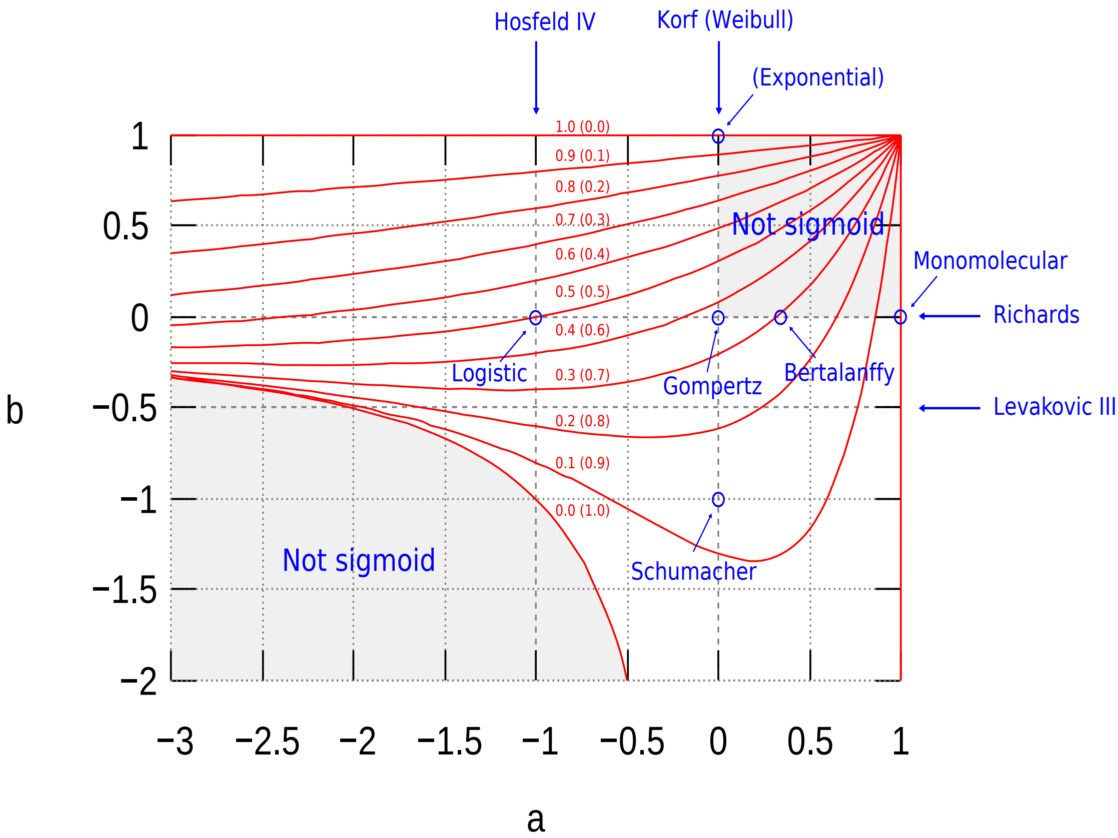

Figure 1: Valid (sigmoid) parameter ranges, and height of the inflection point for the unified growth model (9). Areas with $b \leq 0$ correspond to $G(a, b)$, and those with $b \geq 0$ to $\overline{G}(a, b)$. Items for $\overline{G}(a, b)$ are in parenthesis. Model names according to Zeide (1993). Yield curves with $b = 0$ and $a \leq 0$ do not have an origin at $y = 0$.

In addition to inflection height, some scale-invariant measure of "steepness" (García, 1997), or of mean relative growth (Richards, 1959), could be defined. Then, as it is often done for families of probability distributions, models could be studied further through one-to-one mappings between the two shape parameters and a pair of more meaningful properties (Tadikamalla, 1980; Rodríguez, 1977; García, 1997).

6 SUMMARY AND CONCLUSIONS

Equation (9) includes as special cases a large number of the known univariate growth and yield models. The function B is defined in (1) or (2), and its inverse is (3)–(4). The expression (9) describes the shape of the yield curve, ignoring shifts, scaling, and reflections; the variables t and y may contain additional location and scale parameters.

It is useful to distinguish two more specific forms of (9). $G(a, b)$ is the same as (9). In $\overline{G}(a, b)$, $1 - y$ is substituted for y, and $-t$ for t, that is, the axes are reversed. In G and $\overline{G}$, only shifts and positive scale factors are acceptable, reflections are excluded. The other alternatives, where only one axis is reversed, can be ignored because they do not produce sigmoids.

Current size ("yield") can be thought as an accumulated growth rate: the yield model is the integral of a differential (growth) equation. The growth rate is the derivative of the yield. The fact that the differential equation that generates a given yield model is not unique has caused some confusion. By differentiation and substitution it is possible to obtain one differential equation depending only on t, one depending only on y, or an infinity of equations containing both variables. The general growth equation in terms of y is given in (13).

$G(a, b)$ is a sigmoid, i. e., has an upper asymptote and an inflection point, when the parameters are in the ranges given in (16). Similarly, $\overline{G}(a, b)$ is a sigmoid within (17). These sigmoid regions, and the height of the inflection point, are indicated in Figure 1. The inflection point can be calculated as (15) for G, and as 1 minus that value for $\overline{G}$.

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If $b = 0$ and $a \leq 0$, the sigmoids do not have an origin, that is, do not have $y = 0$ for a finite t. Examples are the logistic, $G(-1, 0)$, and Gompertz, $G(0, 0)$, which range over the entire t-axis.

The region $0 < a < 1$, $b < 0$, and the region $a < 0$, $0 \leq b < 1$, do not seem to have been previously explored as growth models.

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